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THE QUENCH FRONT REVISITED

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ABSTRACT

The cooling of hot surfaces can be modeled in certain simple cases by a nonlinear eigenvalue problem describing the motion of a steady traveling cooling wave. Earlier work on the mathematical theory, the numerical analysis, and the asymptotics of this problem are reviewed.

1. INTRODUCTION

It was nearly fifteen years ago that I first learned about a very interesting heat flow problem which arises in analysing the safety of nuclear reactors. Some of my colleagues and I worked rather hard on this problem for several years, but our last paper was published in 1981, and as far as I can tell there have been no new results obtained by anyone since then, although our work left many open questions. The first paper that I worked on with Joel Dendy and Blair Swartz [3] was given as an invited lecture at the Royal Irish Academy of Sciences Conference on Numerical Analysis in 1976 which was organized by John Miller, and it seems perfectly fitting for me to look at this problem again for this conference.

The actual physical problem is an extremely complicated one in three-dimensional multi-phase thermal-hydraulics. If there is a loss of coolant in a water-cooled nuclear reactor the neutron production stops automatically, but continuing fission provides a source of heat which raises the temperature of the internal surfaces well above the boiling point of water. Any water injected into the system will immediately turn into steam, which is a very poor conductor of heat. The question is, will enough cooling occur to prevent the temperature from rising to the point at which the structure will be damaged? A full answer must involve modeling the flow of water, steam, and other gases through the reactor core and surrounding pipes and walls.

A very elegant idealization of the problem has been proposed in the engineering literature, [5], [8] and [10]. The temperature distribution in an infinite plate with finite thickness is determined by the heat equation;

$$U_t = U_{xx} + U_{yy}.$$

The plate has thickness one, $0 \leq y \leq 1$, it is infinite in extent, $-\infty \leq x \leq \infty$, and the distribution is assumed independent of the third spatial variable. At $x = -\infty$ the plate is cold,

$$U(-\infty, y) = 0,$$

while at the other end it is hot,

$$U(\infty, y) = 1.$$

One surface is taken to be insulated,

$$U_y(x, y)_{y=0} = 0.$$

The boundary condition at the cooling surface is given by

$$U_y(x, y)_{y=1} = -g(U),$$

where $g(U) = 0$ if $U > U_c$, and $g(U) = Bf(U)$ if $U \leq U_c$. U_c is the critical temperature above which no cooling occurs and is assumed to be known, $B > 0$, and $f(U) \geq 0$.

The engineers proposed looking for a traveling wave solution of this boundary value problem. That is, if

$$U(x, y, t) = u(x - vt, y),$$

and if

$$s = x - vt,$$

then

$$u_{ss} + u_{yy} + vu_s = 0. \quad (1.1)$$

The boundary conditions are essentially the same;

$$u(-\infty, y) = 0, \quad u(\infty, y) = 1, \quad (1.2)$$

and,

$$u_y(s, y)_{y=0} = 0, \quad (1.3)$$

except that now we can fix the position of the critical temperature in the moving coordinate system. Thus, we take

$$u(0, 1) = U_c \quad (1.4)$$

and

$$u_y(s, y)_{y=1} = 0, \text{ for } s > 0, \quad (1.5)$$

and

$$u_y(s, y)_{y=1} = -Bf(u(s, 1)) \text{ for } s \leq 0. \quad (1.6)$$

The Quench Front Problem is to find $u(s, y)$, positive, and monotone in s , and the velocity v , satisfying the above. There are three issues to discuss: Existence, numerical methods, and asymptotics.

2. EXISTENCE AND GENERALIZATION

In the special case $f(u) = u$ a proof of the existence of the solution was obtained by Keller and Caflisch [1] using Wiener-Hopf. It is shown in that reference that the velocity v satisfies the following. Let the quantities λ_k , α_k , β_k , $k \geq 1$, be defined by

$$\begin{aligned} \sqrt{\lambda_k} \tan \sqrt{\lambda_k} &= B, \quad 0 < \lambda_1 < \pi < \lambda_2 < 2\pi < \dots, \\ \alpha_k(p) &= \sqrt{1 + 4p\lambda_k}, \end{aligned}$$

$$\beta_k(v) = \sqrt{1 + 4p\pi^2(k-1)^2}.$$

Then v is the solution of

$$\prod_{k=1}^{\infty} \frac{1 + \beta_k(v)}{1 + \alpha_k(v)} = U_c. \quad (2.1)$$

Also assuming that $f(u) = u$, but using separation of variables, Laquer and Wendroff [7] obtained upper and lower bounds for v . Indeed, if

$$H_n(p) = \prod_{k=1}^n \frac{1 + \beta_k(p)}{1 + \alpha_k(p)}, \quad (2.2)$$

$$G_n(p) = \frac{1}{2pB} \frac{\prod_{k=1}^n (\alpha_k(p) - 1)}{\prod_{k=2}^n (\beta_k(p) - 1)}, \quad (2.3)$$

and p_n and q_n are defined by

$$G_n(p_n) = U_c, \quad H_n(q_n) = U_c, \quad (2.4)$$

then,

$$0 < p_2 < p_3 < \dots < q_2 < q_1.$$

In that same paper Laquer and Wendroff also attempted to apply the method of lines. The problem was discretized in the variable y , leaving a system of ordinary differential equations in s . Even for $f(u) = u$ the solution of this system is difficult, although the structure is similar to that for the method of separation of variables. The difficulty lies in the fact that the λ_k defined above now become the eigenvalues λ_k^n of an n -th order matrix corresponding to the discretization of $\partial^2 u / \partial y^2$ along with the boundary conditions. In order to establish bounds for the velocity we needed two results, the first being a purely combinatorial one which we did not succeed in proving, while the second is the following curious fact about the elementary symmetric functions. For $x = (x_1, x_2, \dots, x_n)$ let $e_j(x)$ be the j -th elementary symmetric function of x , that is,

$$e_j(x) = \sum_{1 \leq i_1 < \dots < i_j \leq n} x_{i_1} \dots x_{i_j}.$$

Suppose that $x_i \geq 0$, $y_i \geq 0$, $i = 1, \dots, n$. Then if $e_j(x) \leq e_j(y)$, $j = 1, \dots, n$, then $e_j(x^t) \leq e_j(y^t)$, for $0 < t \leq 1$, where $x^t = (x_1^t, \dots, x_n^t)$. This is proved in [6].

Unlike the method of separation of variables or Wiener-Hopf, the method of lines has the potential for providing theoretical results in the case of nonlinear $f(u)$. It was stated in [7] that a paper by Nicolaenko and Wendroff would appear which would prove an existence theorem for more general $f(u)$, but this paper was never finished and the question of existence is still open. However, in [7] we do show how the maximum principle can be used to obtain positivity and monotonicity results for the method of lines solutions, unfortunately only for $f(u) = u$.

Extending the method of separation of variables to the case of a circular cylinder seems likely to be straightforward. It would also be interesting to have an existence theorem for a cylinder of arbitrary cross-section.

3. A NUMERICAL METHOD

Numerical solution of the quench front problem is difficult. It has the form of a non-linear eigenvalue problem on an infinite domain with a discontinuous boundary condition. In addition, we would like a numerical method that is well-behaved with respect to the parameter B . For large values of B there will be large gradients near the point $(0,1)$ so that some kind of adaptive grid is essential. Dendy and Wendroff [4] devised a variant of the isotherm migration method different from the technique described in [2]. The basic idea is that if $u(s,y)$ is monotone in s then the equation

$$u = u(s,y)$$

can be inverted to express s as a function of u and y ,

$$s = s(u,y),$$

thereby mapping the infinite domain into a finite one. Rather than deal with s as a dependent variable, we proposed an equivalent system of first order equations; namely, introducing new functions

$$w(u,y) = \frac{\partial u}{\partial s}(s(u,y),y), \quad (3.1)$$

and

$$z(u,y) = \frac{\partial u}{\partial y}(s(u,y),y), \quad (3.2)$$

the equations are

$$ww_u + vw + zz_u + z_y = 0, \quad (3.3)$$

and

$$zw_u - wz_u + w_y = 0. \quad (3.4)$$

The boundary conditions become linear; namely,

$$w(0,y) = w(1,y) = 0, \quad (3.5)$$

$$z(u,0) = 0, \quad (3.6)$$

and

$$z(u,1) = 0 \text{ if } u > U_c, \quad (3.7)$$

$$z(u,1) = -Bf(u) \text{ if } u \leq U_c. \quad (3.8)$$

A staggered grid was used, with w centered on cell edges parallel to the u -axis and z centered on edges parallel to the y -axis. Equal u -intervals and equal y -intervals were used. The boundary layer was resolved by introducing an artificial boundary at $y = d$, $0 < d < 1$, setting $z(u,d) = 0$. We used an estimate for d obtained in [9] and then solved the equations for successively smaller d 's until a sufficiently small change in the solution was observed. Examples of the calculation, and more details can be found in [4].

4. ASYMPTOTICS

The asymptotic limits $B \rightarrow 0$ and $B \rightarrow \infty$, corresponding to low and high heat conduction at the cooling surface, are important and interesting. The small B limit was performed (heuristically) in [8] and [10]. For small B we expect $u(s, y)$ to be nearly independent of y . If we integrate eq. (1.1) with respect to y , then apply the boundary conditions and assume all remaining functions are independent of y , we get

$$u_{ss} + v u_s = g(u).$$

As we did in section 3, interchanging the roles of u and s and introducing $w = du/dx$ we obtain the equation

$$w(u)w'(u) + vw(u) = g(u).$$

Since $g(u) = 0$ for $u > U_c$, the solution of the above equation in that range is $w(u) = v(1 - U_c)$, which then leads to the following boundary value problem:

$$ww' + vw = Bf(u), \quad 0 < u < U_c, \quad (4.1)$$

$$w(0) = 0, \quad (4.2)$$

$$w(U_c) = v(1 - U_c). \quad (4.3)$$

It is easy to see that v and w can be scaled by \sqrt{B} , so that v/\sqrt{B} is independent of B for $B \rightarrow 0$. In the case $f(u) = u$ the exact value of v from eqs. (4.1)-(4.3) is

$$v^2 = \frac{B U_c^2}{1 - U_c}.$$

The large B limit seems more difficult. In [9] I tried to find an ordinary differential equation which would be valid for large B . The idea, similar to the method above, is to assume a functional form of the solution and substitute that into a derived integral relation. Crank [2] calls this Goodman's method, but D. Cohen (private communication) informed me that it should be called the Karman-Pohlhausen method. To begin, note that scaling according to

$$s' = Bs, \quad y' = By, \quad v' = v/B,$$

and then dropping the primes the problem remains the same, except that eq. (1.3) is replaced by

$$u_y(s, y)_{y=-B} = 0 \quad (4.4)$$

and eq. (1.6) becomes

$$u_y(s, y)_{y=1} = -f(u(s, 1)) \quad \text{for } s \leq 0. \quad (4.5)$$

B now only appears as the thickness of the slab, which we will let approach infinity. We can also shift y so that now $-\infty < y \leq 0$ and the boundary conditions in y are

$$u_y(s, y)_{y=-\infty} = 0$$

and

$$u_y(s, y)_{y=0} = -f(u(s, 1)) \quad \text{for } s \leq 0.$$

We now suppose that in an interval $-\Delta \leq y \leq 0$ that the temperature is quadratic in y , satisfying

$$u(s, -\Delta) = 1, \quad u_y(s, y)_{y=-\Delta} = 0,$$

and, for $s \leq 0$,

$$u_y(s, y)_{y=0} = -f(u(s, 0)).$$

Introducing $\phi(s) = u(s, 0)$, the quadratic is

$$u(s, y) = \phi(s) - f(\phi)y - \frac{[f(\phi)]^2 y^2}{4(1-\phi)}, \quad (4.6)$$

and

$$\Delta = \frac{2(1-\phi)}{f(\phi)}. \quad (4.7)$$

The integral relation is

$$\int_{-\Delta}^0 [vu_s(s, y) + u_{ss}(s, y)] dy = f(\phi(s)),$$

from which we then obtain

$$\phi'' + v\phi' - (\phi')^2 \alpha(\phi) = F(\phi),$$

where α and F are complicated but explicitly defined expressions in ϕ , f , f' , and f'' . A similar analysis was used to obtain an equation for $s > 0$ which could be solved exactly to obtain a condition at $s = 0$. Making the now familiar change of variables and re-introducing u , the system determining the front velocity vB for large B is

$$ww' + vw - \alpha(u)w^2 = F(u), \quad 0 < u < U_c, \quad (4.8)$$

$$w(U_c) = 2v(1 - U_c), \quad (4.9)$$

$$w(0) = 0. \quad (4.10)$$

Actual calculations of vB show rough agreement with other sources of data.

I can not offer any theoretical justification for this analysis, nor do I know how to obtain corrections for finite B .

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